Laplace Transform

The Laplace transform is an integral transform perhaps second only to the Fourier transform in its utility in solving physical problems. The Laplace transform is particularly useful in solving linear ordinary differential equations such as those arising in the analysis of electronic circuits.

The (unilateral) Laplace transform $\mathcal{L}$ (not to be confused with the Lie derivative, also commonly denoted $\mathcal{L}$) is defined by

$$\mathcal{L}_t[f(t)](s) \equiv \int_0^\infty f(t)e^{-st}\,dt, \tag{1}$$

where $f(t)$ is defined for $t \geq 0$ (Abramowitz and Stegun 1972). The unilateral Laplace transform is almost always what is meant by "the" Laplace transform, although a bilateral Laplace transform is sometimes also defined as

$$\mathcal{L}_t^{(2)}[f(t)](s) = \int_{-\infty}^\infty f(t)e^{-st}\,dt \tag{2}$$

(Oppenheim *et al.* 1997). The unilateral Laplace transform $\mathcal{L}_t[f(t)](s)$ is implemented in *Mathematica* as `LaplaceTransform`[*f[t]*, *t*, *s*].

The inverse Laplace transform is known as the Bromwich integral, sometimes known as the Fourier-Mellin integral (see also the related Duhamel's convolution principle).

A table of several important one-sided Laplace transforms is given below.

f	$\mathcal{L}_t[f(t)](s)$	range
1	$\frac{1}{s}$	$s > 0$
t	$\frac{1}{s^2}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	$n \in \mathbb{Z} > 0$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$	$a > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$s > 0$

$\cosh(at)$	$\frac{s}{s^2-a^2}$	$s > \|a\|$
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$s > \|a\|$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$\delta(t-c)$	e^{-cs}	$c > 0$
$H_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$J_0(t)$	$\frac{1}{\sqrt{s^2+1}}$	
$J_n(at)$	$\frac{(\sqrt{s^2+a^2}-s)^n}{a^n\sqrt{s^2+a^2}}$	$s > 0, n > -1$

In the above table, $J_0(t)$ is the zeroth-order Bessel function of the first kind, $\delta(t)$ is the delta function, and $H_c(t)$ is the Heaviside step function.

The Laplace transform has many important properties. The Laplace transform existence theorem states that, if $f(t)$ is piecewise continuous on every finite interval in $[0, \infty)$ satisfying

$$|f(t)| \leq Me^{at} \tag{3}$$

for all $t \in [0, \infty)$, then $\mathcal{L}_t[f(t)](s)$ exists for all $s > a$. The Laplace transform is also unique, in the sense that, given two functions $F_1(t)$ and $F_2(t)$ with the same transform so that

$$\mathcal{L}_t[F_1(t)](s) = \mathcal{L}_t[F_2(t)](s) \equiv f(s), \tag{4}$$

then Lerch's theorem guarantees that the integral

$$\int_0^a N(t)\,dt = 0 \tag{5}$$

vanishes for all $a > 0$ for a null function defined by

$$N(t) \equiv F_1(t) - F_2(t). \tag{6}$$

The Laplace transform is linear since

$$\mathcal{L}_t[af(t) + bg(t)] = \int_0^\infty [af(t) + bg(t)]e^{-st}\,dt$$

$$\int^\infty \qquad \int^\infty$$

$$
\begin{aligned}
&= a\int_0 \; f e^{-st}\,dt + b\int_0 \; g e^{-st}\,dt \\
&= a\mathcal{L}_t[f(t)] + b\mathcal{L}_t[g(t)]. \qquad (7)
\end{aligned}
$$

The Laplace transform of a convolution is given by

$$\mathcal{L}_t[f(t) * g(t)] = \mathcal{L}_t[f(t)]\mathcal{L}_t[g(t)] \qquad (8)$$

$$\mathcal{L}_t^{-1}[FG] = \mathcal{L}_t^{-1}[F] * \mathcal{L}_t^{-1}[G]. \qquad (9)$$

Now consider differentiation. Let $f(t)$ be continuously differentiable $n-1$ times in $[0,\infty)$. If $|f(t)| \le Me^{at}$, then

$$\mathcal{L}_t[f^{(n)}(t)](s) = s^n\mathcal{L}_t[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0). \qquad (10)$$

This can be proved by integration by parts,

$$
\begin{aligned}
\mathcal{L}_t[f'(t)](s) &= \lim_{a\to\infty}\int_0^a e^{-st}f'(t)\,dt \\
&= \lim_{a\to\infty}\left\{[e^{-st}f(t)]_0^a + s\int_0^a e^{-st}f(t)\,dt\right\} \\
&= \lim_{a\to\infty}\left[e^{-sa}f(a) - f(0) + s\int_0^a e^{-st}f(t)\,dt\right] \\
&= s\mathcal{L}_t[f(t)] - f(0). \qquad (11)
\end{aligned}
$$

Continuing for higher-order derivatives then gives

$$\mathcal{L}_t[f''(t)](s) = s^2\mathcal{L}_t[f(t)](s) - sf(0) - f'(0). \qquad (12)$$

This property can be used to transform differential equations into algebraic equations, a procedure known as the Heaviside calculus, which can then be inverse transformed to obtain the solution. For example, applying the Laplace transform to the equation

$$f''(t) + a_1 f'(t) + a_0 f(t) = 0 \qquad (13)$$

gives

$$
\begin{aligned}
&\{s^2\mathcal{L}_t[f(t)](s) - sf(0) - f'(0)\} + a_1\{s\mathcal{L}_t[f(t)](s) - f(0)\} \\
&\qquad + a_0\mathcal{L}_t[f(t)](s) = 0 \qquad (14)
\end{aligned}
$$

$$\mathcal{L}_t[f(t)](s)\,(s^2 + a_1 s + a_0) - sf(0) - f'(0) - a_1 f(0) = 0, \qquad (15)$$

which can be rearranged to

$$\mathcal{L}_t[f(t)](s) = \frac{sf(0) + f'(0) + a_1 f(0)}{\cdots}. \qquad (16)$$

$$s^2 + a_1 s + a_0$$

If this equation can be inverse Laplace transformed, then the original differential equation is solved.

The Laplace transform satisfied a number of useful properties. Consider exponentiation. If $\mathcal{L}_t[f(t)](s) = F(s)$ for $s > \alpha$ (i.e., $F(s)$ is the inverse Laplace transform of f), then

$\mathcal{L}_t[e^{at} f](s) = F(s-a)$ for $s > a + \alpha$. This follows from

$$\begin{aligned} F(s-a) &= \int_0^\infty f e^{-(s-a)t}\, dt = \int_0^\infty [f(t) e^{at}] e^{-st}\, dt \\ &= \mathcal{L}_t[e^{at} f(t)](s). \end{aligned} \tag{17}$$

The Laplace transform also has nice properties when applied to integrals of functions. If $f(t)$ is piecewise continuous and $|f(t)| \le M e^{at}$, then

$$\mathcal{L}_t\left[\int_0^t f(t')\, dt'\right] = \frac{1}{s} \mathcal{L}_t[f(t)](s). \tag{18}$$

SEE ALSO: Bilateral Laplace Transform, Bromwich Integral, Fourier-Mellin Integral, Fourier Transform, Integral Transform, Laplace-Stieltjes Transform, Operational Mathematics, Unilateral Laplace Transform

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